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A weakly nonlinear theory of continental shelf waves

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In this paper three systems of evolution equations are presented which describe the free propagation of long continental shelf waves in the linear and weakly nonlinear regime. Two different degrees of nonlinearity are considered: for the first the Korteweg–de Vries equation is found to govern the dynamics of the system in the case of a single energy-containing mode (theories by Smith 1972; Grimshaw 1977), whereas, for the second nonlinear range, a nonlinear hyperbolic equation is derived. The nonlinear interactions among shelf-wave modes are also considered: they are modelled through nonlinear coupling terms in the evolution equations. This theory allows the timescale for the development of dispersive and nonlinear effects to be determined for each parameter range. The amplitude ranges corresponding to linear and nonlinear shelf waves are evaluated for the Oregon and the East Australian shelves, and some qualitative conclusions on the importance of nonlinear effects are derived. Finally the case of a shelf with longshore variation in topography is analysed and coupling terms in the evolution equations appear. They account for the scattering of energy between the various modes due to the linear and nonlinear interactions of the wave with the topographic changes.

1. Introduction

Continental shelf waves are subinertial waves confined to the continental shelf and propagating with the coast to the right in the Northern hemisphere. The first theory on shelf waves was proposed by Robinson (1964), who assumed the waves to be so long as to be non-dispersive. For shorter wavelengths the waves become dispersive (Mysak 1968; Buchwald & Adams 1968). Fundamental articles on the generation of long shelf waves are by Adams & Buchwald (1969) and Gill & Schumann (1974). Mysak (1980) reviews recent developments in the research on shelf-wave dynamics.

In most papers the equations are linearized, this being justifiable in many situations owing to the small size of the Rossby number for shelf waves. Nonetheless there are some typically nonlinear problems which have been dealt with in the literature. In a paper by Hsieh & Mysak (1980) the resonant interaction between shelf waves is analysed. A theory of the sideband instability and long-wave resonance for shelf waves is available (Grimshaw, 1977*a*). Nonlinear effects can also lead to solitary-wave solutions. Smith (1972) and Grimshaw (1977*b*) developed two complementary theories where nonlinearity and dispersion balance out exactly, and a nonlinear evolution equation where the dispersive term was modelled through a pseudodifferential operator, and the Korteweg–de Vries equation was found to give the time evolution of the amplitude of the shelf wave.

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However, in these last two treatments only the evolution of a single wave mode is dealt with (i.e. the wave energy is assumed to be contained in only one mode). Therefore the most general initial-value problem, for which the energy is necessarily spread over the various modes, can only be treated by a more general analysis.

In this paper a system of nonlinear evolution equations is proposed, which gives the time evolution of the most general initial condition for free continental shelf waves. Such equations contain nonlinear interaction terms coupling the various modes. Two degrees of nonlinearity are considered. For the first one the equations reduce to the Korteweg–de Vries equation on the assumption of a single shelf-wave mode (Grimshaw 1977*b*). For the second one (stronger nonlinearities) they reduce to a nonlinear hyperbolic equation on the same assumption. A discussion on the importance of nonlinear effects for shelf waves in connection with the results of this treatment is also presented.

Another problem dealt with here is the effect of dispersion and nonlinearities on the scattering of shelf waves by longshore variations in bottom topography. Allen (1976) and Hsueh (1980) studied this kind of scattering in the linear approximation and assuming lack of phase dispersion (very long waves). Under these conditions they found that, although most longshore depth changes provide a scattering of energy between the various modes, a class of bottom topographies (denoted as 'shelf-similar') exists, for which no scattering occurs. Here it is shown that the requirement that shelf waves be linear and non-dispersive is critical for this property to hold. Indeed a system of equations is presented which applies to linear/weakly dispersive and nonlinear shelf waves in the case of a shelf-similar bottom topography, where it appears that the modes are coupled as a consequence of such a longshore variation in topography, so that scattering does occur.

2. Governing equations

The dynamics of unforced, barotropic continental shelf waves is governed by the shallow-water equations (Le Blond & Mysak 1978):

$$\bar{u}_{\bar{t}}+\bar{u}\bar{u}_{\bar{x}}+\bar{v}\bar{u}_{\bar{y}}-f\bar{v} = -g\bar{\zeta}_{\bar{x}}, \tag{1a}$$

$$\bar{v}_{\bar{t}}+\bar{u}\bar{v}_{\bar{x}}+\bar{v}\bar{v}_{\bar{y}}+f\bar{u} = -g\bar{\zeta}_{\bar{y}}, \tag{1b}$$

$$\bar{\zeta}_{\bar{t}}+[(\bar{H}+\bar{\zeta})\,\bar{u}]_{\bar{x}}+[(\bar{H}+\bar{\zeta})\,\bar{v}]_{\bar{y}} = 0, \tag{1c}$$

where f can be taken as constant (Buchwald & Adams 1968). The variables $\bar{u}$ and $\bar{v}$ are the velocity components in the onshore ($\bar{x}$) and alongshore ($\bar{y}$) directions respectively, $\bar{\zeta}$ is the surface elevation and $\bar{H}$ is the undisturbed water depth – see figure 1 (we shall consider cases where $H' > 0$). If T is the timescale (i.e. $\bar{t} = Tt$, where henceforth letters without bars denote the non-dimensionalized variables), we have assumed for shelf waves

$$\epsilon = \frac{1}{fT} \ll 1. \tag{2}$$

As we want to perform an expansion of the surface elevation in powers of ϵ, we pass to non-dimensional variables:

$$\left.\begin{aligned} &\bar{x} = Lx, \quad \bar{y} = L_y y = \frac{L}{\epsilon}y, \\ &\bar{u} = Uu, \quad \bar{v} = Vv = \frac{U}{\epsilon}v, \\ &\bar{\zeta} = N\zeta, \quad \bar{H} = DH. \end{aligned}\right\} \tag{3}$$

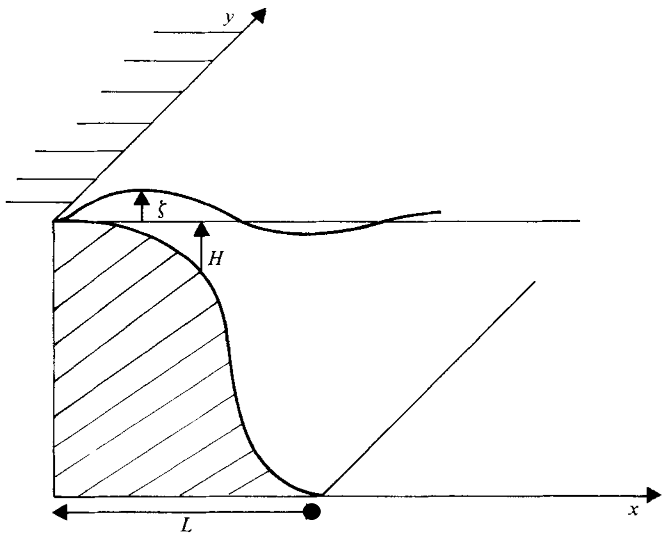

FIGURE 1. Definition of x, y, ζ, H, L.

The different length and velocity scales in the two spatial directions are introduced in order to model the scale anisotropy of shelf waves. Besides, the definition $L_y = L/\epsilon$ implies that we are studying long waves. The onshore lengthscale L is given by the shelf width. Moreover, long shelf waves are known to be generated by the passage of large-scale weather systems (Gill & Schumann 1974) whose dimension is $O(1000\text{ km})$; therefore L_y is of the same order of magnitude. Since the external Rossby deformation radius $R = (gD)^{\frac{1}{2}}/f = O(1000\text{ km})$ for a typical continental shelf, we choose

$$L_y = R.$$

So for (3) we have that ϵ coincides with the divergence parameter:

$$\epsilon = \frac{L}{L_y} = \frac{fL}{(gD)^{\frac{1}{2}}}. \tag{4}$$

This assumption simplifies the analysis considerably. In order to achieve the balance between the $O(1)$ terms in the non-dimensional version of $(1a)$ we find

$$N = \frac{fUL}{\epsilon g} \equiv \epsilon^q D, \tag{5}$$

where the notation $N = \epsilon^q D$ is introduced for the sake of convenience. From (3)–(5) we also see that

$$U = \epsilon^{q-1} fL, \quad V = \epsilon^{q-2} fL. \tag{6}$$

Rewriting (1) in terms of the non-dimensional variables and taking (3)–(6) into account, we get

$$\epsilon^2 u_t + \epsilon^q \boldsymbol{V}\cdot\nabla u - v = -\zeta_x, \tag{7a}$$

$$v_t + \epsilon^{q-2}\boldsymbol{V}\cdot\nabla v + u = -\zeta_y, \tag{7b}$$

$$\epsilon^2 \zeta_t + [(H+\epsilon^q\zeta)\,u]_x + [(H+\epsilon^q\zeta)\,v]_y = 0, \tag{7c}$$

where $V = (u, v)$. Combining $(7a, b)$ in order to express u and v as functions of the ζ-derivatives and substituting them in $(7c)$, we ultimately obtain

$$\epsilon^2\zeta_t + \{(H+\epsilon^q\zeta)\,[-\zeta_y - \zeta_{xt} + \epsilon^2(\zeta_{ytt} + \zeta_{xttt}) + \epsilon^{q-2}(\zeta_y\,\zeta_{xx} + \zeta_{xt}\,\zeta_{xx} - \zeta_x\,\zeta_{xy}) + O(\epsilon^q, \epsilon^{q-2+n})]\}_x + \{(H+\epsilon^q\zeta)\,[\zeta_x - \epsilon^2(\zeta_{yt} + \zeta_{xtt}) + O(\epsilon^4, \epsilon^q)]\}_y = 0,$$
$$n = \min\,[q-2, 2] = 1, 2. \qquad (8)$$

The terms $O(\epsilon^q, \epsilon^{q-2+n}, \epsilon^4)$ contain the velocity components and are always negligible in the cases studied, so (8) is one equation for one unknown (ζ) at the lowest order. The velocity components can be expressed in terms of ζ as follows:

$$\left.\begin{aligned} u &= -\zeta_y - \zeta_{xt} + O(\epsilon^2, \epsilon^{q-2}), \\ v &= \zeta_x + O(\epsilon^2). \end{aligned}\right\} \qquad (9)$$

3. The evolution equations for the wave amplitude

In this section the case of a uniform $(H_y = 0)$ and non-uniform shelf are dealt with separately.

3.1. *Waves on uniform shelves*

In order to derive the final evolution equations from (8) it is convenient to perform a Galilean transformation:

$$y' = y + ct, \quad x' \equiv x,$$

where c is the phase speed of non-dispersive waves. In the following discussion we shall limit ourselves to a single energy-containing mode, the general case being studied later. In this moving reference frame the timescale T' will be different from T. Let us define $T' = T/\epsilon^s$, where s is still unknown. In this new frame of reference the appropriate time variable will thus be $\tau = \epsilon^s t$, so we have (the primes have been dropped):

$$\zeta_t = \epsilon^s\zeta_\tau + c\zeta_y.$$

Let us also expand ζ as follows:

$$\zeta = \phi(x)\,A(y, t) + \epsilon^s\Phi^{(1)}(x, y, t) + \dots. \qquad (10)$$

With these positions (8) at zeroth order in ϵ gives

$$(H\phi')' = -\frac{H'}{c}\phi. \qquad (11)$$

The associated boundary conditions are

$$\left.\begin{aligned} &Hu = 0 \quad (x = 0) \\ &\lim_{x\to\infty} \zeta = 0 \end{aligned}\right\}, \quad \text{if} \quad H(0) = 0 \quad \Rightarrow \quad \left\{\begin{aligned} &|\phi'(0)| < \mathscr{C} = \text{const}, \\ &\lim_{x\to\infty} \phi = 0. \end{aligned}\right.$$

Equation (11) with these boundary conditions constitutes a classical Sturm–Liouville problem (Morse & Feshbach 1953). Therefore the $\phi_i(x)$ form an orthogonal (with H' as density function) and complete set of eigenfunctions with positive eigenvalues c^{-1} (indeed shelf waves are right-bounded in the Northern Hemisphere $-f > 0-$). Solutions of this eigenvalue problem are obtained, for particular depth profiles, by Mysak (1968). Finally we shall deal with normalized eigenfunctions, i.e.

$$\int_0^\infty H'\phi_i\,\phi_k\,\mathrm{d}x = \delta_{ik}.$$

Taking now (10) and (11) into account and expanding $\Phi_i^{(1)}$ as

$$\Phi_i^{(1)}(x,y,t) = \sum_j p_{ij}(y,t)\,\phi_j(x),$$

(8) gives

$$\epsilon^s\left[\sum_j p_{ijy}\left(\frac{c_i}{c_j}-1\right)H'\phi_j + \frac{H'}{c_i}\phi_i A_{i\tau}\right] + \epsilon^2[c_i\phi_i A_{iy} - c_i H\phi_i A_{iyyy}]$$
$$+\epsilon^{q-2}[H(\phi_i\phi_i''+c_i\phi_i'\phi_i''-\phi_i'^2)]'A_i A_{iy} + D = 0, \quad (12)$$

where $D = O(\epsilon^q, \epsilon^{q-2+s}, \epsilon^{q-2+n}, \epsilon^{2+s}, \epsilon^{2s}, \epsilon^4)$. If we multiply by ϕ_i and integrate over x we find

$$\epsilon^s A_{i\tau} + \epsilon^2(\alpha_i A_{iy} + \beta_i A_{iyyy}) + \epsilon^{q-2}\gamma_i A_i A_{iy} + \int_0^\infty D\phi_i\,\mathrm{d}x = 0, \quad (13)$$

where α_i, β_i and γ_i are defined in the Appendix. It is now clear that s must be

$$s = \min[2, q-2] \equiv n,$$

and then $D \ll \max[\epsilon^2, \epsilon^{q-2}]$. Therefore, neglecting higher-order terms and coming back to the frame at rest, from (13) we finally get

$$\epsilon^{-2}(A_{it} - \tilde{c}_i A_{iy}) + \beta_i A_{iyyy} = 0 \quad (q \geqslant 5;\ s = 2), \quad (14)$$

$$\epsilon^{-2}(A_{it} - \tilde{c}_i A_{iy}) + \beta_i A_{iyyy} + \gamma_i A_i A_{iy} = 0 \quad (q = 4;\ s = 2), \quad (15)$$

$$\epsilon^{-1}(A_{it} - c_i A_{iy}) + \gamma_i A_i A_{iy} = 0 \quad (q = 3;\ s = 1), \quad (16)$$

where $\tilde{c}_i = c_i - \epsilon^2\alpha_i$.

Equation (14) describes linear weakly dispersive waves; the corresponding dispersion relation is (in dimensional form):

$$\bar{\omega}_i = fL(\tilde{c}_i\bar{K} + \epsilon^2\beta_i R^2\bar{K}^3),$$

which holds for wavenumbers such that $\bar{K} \leqslant O(R^{-1})$. Equation (15) is the well-known Korteweg–de Vries equation, in which the exact balance between nonlinearity (which tends to steepen the wave) and dispersion (which tends to spread it) leads to soliton solutions (e.g. Gardner *et al.* 1974). Moreover, an arbitrary initial condition evolves into a set of solitons plus a small dispersive tail. Solitonic behaviour is therefore expected for shelf waves whose energy is concentrated in only one mode and for the appropriate parameter range ($q = 4$). This result was previously obtained by Grimshaw (1977*b*). Finally, (16) is a hyperbolic equation solvable implicitly by the method of characteristics (Whitham 1974) which leads to the progressive steepening of the wave and eventually to its breaking.

So far we have made use of the rather unrealistic hypothesis (10), whereas, for the more general initial-value problem, ζ needs to be defined as follows:

$$\zeta = \sum_k \zeta_k = \sum_k[\phi_k A_k + \epsilon^s\Phi_k^{(1)} + \ldots]. \quad (17)$$

For $q \geqslant 5$ each A_k still evolves according to (14) owing to the linearity of the wave, but for $q = 3, 4$ the various wave modes will interact nonlinearly. Performing a change of variable for each mode,

$$\zeta_{kt} = \epsilon^s\zeta_{k\tau} + c_k\zeta_{ky},$$

substituting (17) in (8) and performing the same calculations as before, one gets, for $q = 4$,

$$\epsilon^{-2}(A_{it} - \tilde{c}_i A_{iy}) + \beta_i A_{iyyy} + \gamma_i A_i A_{iy} + \sum_{\substack{j,k \\ j \neq k}} \gamma_{jk}^i A_j A_{ky}$$
$$= -\sum_{\substack{k \\ k \neq i}} \{p_{kiy}(c_k - c_i) + \alpha_{ik} A_{ky} + \beta_{ik} A_{kyyy} + \gamma_{kk}^i A_k A_{ky}\}, \quad (18)$$

where the coefficients $\alpha_{ik}, \beta_{ik}, \gamma_{jk}^i$ are defined in the Appendix. But we know from the preceding analysis that if at $t = 0$ there is only one energy-containing mode, say l, the remaining modes have $A_k = 0$ $(k \neq l)$ as solution, and this is only possible if the right-hand side of (18) is zero (this gives a relation between the unknown functions p_{ki} and the A_k). Therefore the following set of coupled evolution equations results:

$$\epsilon^{-2}(A_{it} - \tilde{c}_i A_{iy}) + \beta_i A_{iyyy} + \gamma_i A_i A_{iy} + \sum_{\substack{j,k \\ j \neq k}} \gamma_{jk}^i A_j A_{ky} = 0. \tag{19}$$

From (19) we see that, for the initial conditions of the form $A_k(t = 0) = 0$ $(k \neq l)$, the solution is given by $A_k(t) = 0$ $(k \neq l)$, whereas the variable A_l satisfies (15). If, on the other hand, at the initial time two or more modes contain some energy, all the remaining modes will be forced by the nonlinear interaction terms

$$\sum_{\substack{j,k \\ j \neq k}} \gamma_{jk}^i A_j A_{ky}.$$

So scattering of energy occurs among modes due to nonlinear interactions. A similar set of equations can be found for $q = 3$ as a generalization of (16):

$$\epsilon^{-1}(A_{it} - c_i A_{iy}) + \gamma_i A_i A_{iy} + \sum_{\substack{j,k \\ j \neq k}} \gamma_{jk}^i A_j A_{ky} = 0. \tag{20}$$

3.2. *Waves on weakly non-uniform shelves*

In the following we investigate the changes occurring in the analysis carried out in the preceding paragraph if a small-amplitude longshore variation in bottom topography is present.

Suppose the function $\xi(x,y)$ is defined so that the lines $\xi(x,y) = \text{const}$ are depth contours (we would have $\xi \equiv x$ for a uniform shelf). This definition implies $H(x,y) \equiv H(\xi)$. In the present case, where H depends on both spatial coordinates, it is convenient to pass to the new variables (ξ, η) where $\eta \equiv y$. We have

$$\zeta_y = \zeta_\eta + \xi_y \zeta_\xi, \quad \zeta_x = \xi_x \zeta_\xi,$$

$$H_y = H'\xi_y, \quad H_x = H'\xi_x \quad (H' \equiv H_\xi).$$

Let us start by considering the linear case $(q \geqslant 5)$. Performing the transformation

$$\eta' = \eta + c_k t, \quad \xi' \equiv \xi, \quad \tau = \epsilon^2 t,$$

$$\zeta_{kt} = \epsilon^2 \zeta_{k\tau} + c_k \zeta_{k\eta}$$

(the primes have been dropped), and substituting

$$\zeta = \sum_k [\phi_k(\xi) A_k(\eta, t) + \epsilon^2 \Phi_k^{(1)}(\xi, \eta, t) + \ldots]$$

in (8), we get at lowest order

$$\frac{1}{c_k} H'\xi_x \phi_k + (H'\xi_x^2 + H\xi_{xx}) \phi_k' + H\xi_x^2 \phi_k'' = 0, \tag{21}$$

and at the second order (with (21) taken into account)

$$\begin{aligned}\sum_k \Big\{ &-c_k \left[\frac{1}{c_k} H'\xi_x \Phi_{k\eta}^{(1)} + (H'\xi_x^2 + H\xi_{xx}) \Phi_{k\eta\xi}^{(1)} + H\xi_x^2 \Phi_{k\eta\xi\xi}^{(1)}\right] \\ &+ \left[\frac{1}{c_k} H'\xi_x \phi_k\right] A_{k\tau} + [c_k \phi_k - c_k H(\xi_y^2 \phi_k'' + \xi_{yy} \phi_k') - c_k H'\xi_y^2 \phi_k'] A_{k\eta} \\ &+ [-c_k \xi_y (2H\phi_k' + H'\phi_k)] A_{k\eta\eta} + [-c_k H\phi_k] A_{k\eta\eta\eta} \Big\} + O(\epsilon^3) = 0. \end{aligned} \tag{22}$$

Equations (21) and (22) reduce to (11) and (12) ($q \geqslant 5, s = 2$, single mode) respectively, in the limit $\xi \equiv x$.

At this stage we follow Allen (1976) in defining a small-amplitude longshore variation in bottom topography:

$$\xi = x + \theta h(x, y), \tag{23}$$

where θ is a small parameter, $\theta \ll 1$. This gives

$$\xi_x \approx 1 + \theta h_x \approx 1 + \theta h_\xi,$$

$$\xi_y \approx \theta h_y \approx \theta h_\eta.$$

Let us expand c_k and ϕ_k in powers of θ:

$$\phi_k = \phi_{0k} + \theta\phi_{1k} + \dots,$$

$$c_k = c_{0k} + \theta c_{1k} + \dots.$$

At lowest order in θ, (21) gives

$$(H\phi'_{0k})' = -\frac{H'}{c_{0k}}\phi_{0k}, \tag{24}$$

with the same boundary conditions as (11). Equation (24) is identical with (11); therefore the $\phi_{0k}(\xi)$ have the same functional form as the $\phi_k(x)$ of the $\theta = 0$ theory, that is (at lowest order) the eigenfunctions $\phi_k(\xi)$ (which we may denote as 'local' because they depend weakly on y) adjust to the depth variation along the coast. Defining

$$\phi_{1k} = \sum_j \pi_{kj}\phi_{0j},$$

we obtain from (21), at first order in θ,

$$\pi_{kj} = \left(\frac{c_{0k}c_{0j}}{c_{0k}-c_{0j}}\right) g_{kj} \quad (k \neq j), \quad \pi_{kk} = 0,$$

$$c_{1k} = c_{0k}^2 g_{kk}, \quad g_{kj} = \int_0^\infty (h_\xi H\phi'_{0k})_\xi \phi_{0j}\,\mathrm{d}\xi.$$

Let us now examine the time evolution of ζ, limiting ourselves to bottom topographies whose deformation offshore is linear in x, i.e.

$$h_{\xi\xi} = 0 \quad \Rightarrow \quad \xi = x[1 + \theta r(y)]. \tag{25}$$

This kind of topography is denoted as 'shelf-similar' by Hsueh (1980) and does not give rise to scattering among modes for linear and non-dispersive shelf waves. However, in the following treatment we shall see that both weak dispersion and weak nonlinearity lead to mode coupling, and scattering will occur as a result. Thanks to (25) we have

$$c_{1k} = -c_{0k}h_\xi, \quad g_{kj} = -\frac{h_\xi}{c_{0k}}\delta_{kj}, \quad \phi_{jk} = 0 \quad (j \geqslant 1, \forall\, k).$$

Taking these relations into account one gets from (22)

$$\epsilon^{-2}(1 + 2\theta h_\xi)(A_{it} - \tilde{c}_i A_{i\eta}) + \beta_i A_{i\eta\eta\eta} = \theta \sum_k [d_{ik}^{(1)} A_{k\eta} + d_{ik}^{(2)} A_{k\eta\eta} + d_{ik}^{(3)} A_{k\eta\eta\eta}], \tag{26}$$

where the coefficients $d^{(1)}$, $d^{(2)}$ and $d^{(3)}$ are defined in the Appendix. This is a system of weakly coupled linear partial differential equations where the coupling coefficients depend on η. It reduces to (14) in the uniform-shelf case. The presenee of the amplitudes A_k of all the modes in the right-hand side of (26) provides the scattering of energy into different modes due to the interaction of the wave with the topography.

The equations generalizing (19) and (20) can be obtained in a similar way. They are

$$\epsilon^{-2}(1+2\theta h_\xi)(A_{it}-\tilde{c}_i A_{i\eta})+\beta_i A_{i\eta\eta\eta}+\gamma_i A_i A_{i\eta}+\sum_{\substack{j,k\\ j\neq k}}\gamma^i_{jk} A_j A_{k\eta}$$
$$=\theta\sum_k [d^{(1)}_{ik} A_{k\eta}+d^{(2)}_{ik} A_{k\eta\eta}+d^{(3)}_{ik} A_{k\eta\eta\eta}+\sum_j (d^{(4)}_{ijk} A_j A_{k\eta}+d^{(5)}_{ijk} A_j A_k)] \quad (27)$$

for $q = 4$ (see the Appendix for $d^{(4)}$ and $d^{(5)}$), and

$$\epsilon^{-1}(1+2\theta h_\xi)(A_{it}-c_i A_{i\eta})+\gamma_i A_i A_{i\eta}+\sum_{\substack{j,k\\ j\neq k}}\gamma^i_{jk} A_j A_{k\eta}=\theta\sum_{j,k}[d^{(4)}_{ijk} A_j A_{k\eta}+d^{(5)}_{ijk} A_j A_k] \quad (28)$$

for $q = 3$. They reduce to (19) and (20) respectively in the limit $\theta\to 0$.

4. Discussion

In this section the orders of magnitude of the quantities appearing in the present analysis will be evaluated in some realistic examples, and some physical conclusions on the importance of nonlinear effects will be derived.

Taking the width of the shelf as $L = 5\times 10^4$ m and the depth as $D = 250$ m (a typical value for shelves with a steep continental slope) and $f = 10^{-4}\ \mathrm{s}^{-1}$ we have $R = 500$ km and $\epsilon = 10^{-1}$. Relations (5) and (6) give $N = 0.25$ cm/s as the maximum order of magnitude of the free-surface elevation for linear waves ($q = 5$) with a corresponding $V = 0.5$ cm/s for the amplitude of the longshore current fluctuations. The balance between nonlinearity and dispersion ($q = 4$) is achieved for $N = 2.5$ cm and $V = 5$ cm/s, whereas the wave front steepens ($q = 3$) above the minimum values $N = 25$ cm and $V = 50$ cm/s. The timescale for the development of dispersive and weakly nonlinear effects is $T'_{\mathrm{weak}} = T/\epsilon^2 \approx 10^7$ s ≈ 100 days, whereas the steepening effects develop on a much shorter timescale: for $q = 3$ it is $T'_{\mathrm{strong}} = T/\epsilon \approx 10^6$ s ≈ 10 days.

From this we see that the purely dispersive and combined weakly dispersive–weakly nonlinear effects are so weak for long continental shelf waves as to be negligible in most situations, at least along terrestrial coasts (this had already been noted by Smith 1972; Grimshaw 1977*b*). Indeed, if one considers the gravest mode and assumes $c_1 \approx 0.5$, the 'life' of a free shelf wave for a coast of length $l \approx 2000$ km can be $T_{\mathrm{life}} = l/\bar{c} = l/fLc \approx 10$ days $\ll T'_{\mathrm{weak}}$.

So, when they apply, (14) and (19) are basically equivalent to the simple linear, non-dispersive equation

$$A_{it}-\tilde{c}_i A_{iy} = 0.$$

In contrast, nonlinear effects of the hyperbolic kind ($q = 3$) may lead to a significant change in the shape of a free long shelf wave along a real coast, because, for instance in the present example, $T_{\mathrm{life}} \approx T'_{\mathrm{strong}} \approx 10$ days; moreover, higher modes ($k = 2, 3, \ldots$) have smaller phase speeds and thus longer lives; therefore for them these nonlinear effects are even more important. So, provided that $N_{\mathrm{strong}} = \epsilon^3 D$ and $V_{\mathrm{strong}} = fL\epsilon$ represent good length and velocity scales for non-dimensionalising $\bar{\zeta}$ and $\bar{v}$ respectively, the combined effects of the steepening of the wave front and of the scattering of energy among the modes due to nonlinear interactions are expected to develop in a sufficiently short time for them to be observed.

It also appears that the experimentally observed values of sea level and current fluctuation amplitudes (roughly $O(10 \text{ cm})$ and $O(10 \text{ cm/s})$ respectively) are not far from the theoretical values for 'strong' waves ($q = 3$), so that such waves can actually be generated. But, in order to analyse this point more carefully, some realistic cases have to be considered.

Let us consider two coasts along which continental shelf waves have been clearly identified, namely the Oregon and the East Australian coasts (e.g. Hsieh 1982; Hamon 1962; Robinson 1964). The Buchwald–Adams (1968) exponential shelf profile $H = H_0 e^{bx}$ can be fitted to the real depth profiles off both the Oregon and the East Australian coasts:

$$\text{Oregon:} \quad \bar{H} = 104\, e^{29 \times 10^{-3} \bar{x}}, \quad \bar{H}_{\max} = 2840 \text{ m} \quad (L = 112 \text{ km});$$

$$\text{East Australia:} \quad \bar{H} = 23\, e^{67 \times 10^{-3} \bar{x}}, \quad \bar{H}_{\max} = 5090 \text{ m} \quad (L = 80 \text{ km});$$

where $\bar{H}$ and $\bar{x}$ are expressed in kilometres. D may be defined through $\bar{H}(\bar{x})$ by

$$D = \frac{1}{L} \int_0^L \bar{H}(\bar{x})\, \mathrm{d}\bar{x}$$

(the fact that $\bar{H}(0) \neq 0$ has obviously no importance in this example). Thus we have

$$\text{Oregon:} \quad \begin{cases} D = 828 \text{ m}, \quad L = 112 \text{ km}, \quad f = 10^{-4}, \text{ s}^{-1}, \\ R = 900 \text{ km}, \quad \epsilon = 1.2 \times 10^{-1}, \\ T = 0.96 \text{ days}, \quad fL = 11.2 \text{ m/s}; \end{cases}$$

$$\text{East Australia:} \quad \begin{cases} D = 937 \text{ m}, \quad L = 80 \text{ km}, \quad f = 0.83 \times 10^{-4}, \text{ s}^{-1}, \\ R = 1150 \text{ km}, \quad \epsilon = 7 \times 10^{-2}, \\ T = 2 \text{ days}, \quad fL = 6.6 \text{ m/s}. \end{cases}$$

The values of V and T' in the various cases are given in table 1, where the evolution equations corresponding to each case are also reported.

From table 1 it can be seen that although T'_{strong} for Oregon is very small (only 8 days), the corresponding V_{strong} is so high (130 cm/s versus the observed value of about 20 cm/s) that it seems unlikely that one can ever observe nonlinear effects off the Oregon coast, except for the resonant interaction between shelf waves, which does indeed play a significant role on this shelf (Hsieh & Mysak 1980). However, were strongly nonlinear shelf waves generated, their profile would very rapidly change, following the evolution given by (20).

In fact, along the East Australian coast the value $V_{\text{strong}} = 46$ cm/s is likely to be observed (it must be borne in mind that V gives only the order of magnitude of the current oscillation, not its real amplitude). There are still no current data available for shelf waves on this shelf, but we can compare this value to the theoretical ones obtained by Gill & Schumann (1974) in a fundamental article on the generation of long shelf waves by wind stress. It was shown that in a linear non-dispersive model applied to the East Australian coast, a wind stress of 1 dyn/cm^2 generates shelf waves with current amplitudes of 22 cm/s and 17 cm/s for the first two modes respectively (the energy of the remaining modes being negligible) and one can see that the ratio between V_{strong} and these values (≈ 2) is much smaller than the ratio between such values and V_{weak} (≈ 7).

Therefore these two shelf waves generated by the wind stress according to Gill–Schumann's theory are then subjected to a free evolution that would exhibit the

q	s	Evolution equation	Oregon shelf V (cm/s)	Oregon shelf T' (days)	East Australian shelf V (cm/s)	East Australian shelf T' (days)
$\geqslant 5$	2	$\epsilon^{-2}(A_{it}-\tilde{c}_i A_{iy})+\beta_i A_{iyyy}=0$	1.9 ($q=5$)	66 ($q=5$)	0.22 ($q=5$)	408 ($q=5$)
4	2	$\epsilon^{-2}(A_{it}-\tilde{c}_i A_{iy})+\beta_i A_{iyyy} + \gamma_i A_i A_{iy} + \sum_{j,k;\, j\neq k} \gamma^i_{jk} A_j A_{ky} = 0$	16	66	3.2	408
3	1	$\epsilon^{-1}(A_{it}-c_i A_{iy}) + \gamma_i A_i A_{iy} + \sum_{j,k;\, j\neq k} \gamma^i_{jk} A_j A_{ky} = 0$	130	8	46	28

TABLE 1. For each value of q, the value of s, the corresponding evolution equation for the $A_i(y,t)$ (in the case $H_y = 0$) and the values of V and T' for both the Oregon and the East Australian shelves are given.

steepening of their fronts and their mutual nonlinear interaction, instead of travelling undisturbed as a linear nondispersive theory would forecast. The time T'_{strong} is not as small as in the previous case but it is still $T'_{\text{strong}} \approx \frac{1}{3}T_{\text{life}}$ for the gravest mode and $T'_{\text{strong}} \approx T_{\text{life}}$ for the second mode, so that nonlinear effects should not be neglected when modelling the free evolution of the waves under consideration.

5. Conclusions

We have studied the free propagation of long barotropic continental shelf waves over uniform and weakly non-uniform shelves. If N (the lengthscale used to non-dimensionalize the surface displacement $\bar{\zeta}$) is expressed as $N = \epsilon^q D$ (where ϵ is the divergence parameter and D is the mean depth) we find that, if $q \geqslant 5$, the wave amplitudes satisfy (14) for uniform shelves or (26) for weakly non-uniform shelves, if $q = 4$ the amplitudes satisfy (19) or (27), and finally if $q = 3$ they satisfy (20) or (28).

The case $q = 3$ (which we have called 'strongly' nonlinear) is the most interesting, because the corresponding dynamics cannot be approximated by a linear non-dispersive model (as is the case for linear or weakly nonlinear waves), so that (20) and (28) may have a practical application. This case is also interesting as far as the interpretation of observations is concerned because in some significant cases (for instance off the East Australian coast) shelf waves may be generated with $q \approx 3$.

Finally we note that the model equations derived in the various cases can be used in order to simulate numerically the dynamics of free long shelf waves (the author is currently engaged in this numerical work). This would permit the scattering processes due to nonlinear interactions and induced by the longshore variations in bottom topography to be studied. The equations valid for weakly non-uniform shelves might be particularly useful in modelling more realistic situations. In all cases, comparatively few modes (say the first two or three) should be sufficient in order to obtain a good description of the physical processes involved.

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Appendix

$$\alpha_i = c_i^2 \int \phi_i^2 \, \mathrm{d}x, \quad \beta_i = -c_i^2 \int H\phi_i^2 \, \mathrm{d}x,$$

$$\gamma_i = c_i \int [H(-\phi_i'^2 + \phi_i \phi_i'' + c_i \phi_i' \phi_i'')]_x \phi_i \, \mathrm{d}x,$$

$$\alpha_{ik} = c_i c_k \int \phi_i \phi_k \, \mathrm{d}x, \quad \beta_{ik} = -c_i c_k \int H\phi_i \phi_k \, \mathrm{d}x,$$

$$\gamma_{jk}^i = c_i \int [H(-\phi_j' \phi_k' + \phi_j'' \phi_k + c_k \phi_j'' \phi_k')]_x \phi_i \, \mathrm{d}x,$$

$$d_{ik}^{(1)} = \begin{cases} c_{0i} c_{0k} \int (h_{\eta\eta} H\phi_{0k}' + 2h_\xi \phi_{0k}) \phi_{0i} \, \mathrm{d}\xi & (i \neq k), \\ c_{0i}^2 \int (h_{\eta\eta} H\phi_{0i}' + h_\xi \phi_{0i}) \phi_{0i} \, \mathrm{d}\xi, & (i = k), \end{cases}$$

$$d_{ik}^{(2)} = c_{0i} c_{0k} \int h_\eta (2H\phi_{0k}' + H'\phi_{0k}) \phi_{0i} \, \mathrm{d}\xi,$$

$$d_{ik}^{(3)} = \begin{cases} -2c_{0i} c_{0k} h_\xi \int H\phi_{0i} \phi_{0k} \, \mathrm{d}\xi & (i \neq k), \\ -c_{0i}^2 h_\xi \int H\phi_{0i}^2 \, \mathrm{d}\xi, & (i = k), \end{cases}$$

$$d_{ijk}^{(4)} = \begin{cases} (\delta_{jk} - 3) h_\xi \gamma_{jk}^i & ((j,k) \neq (i,i)), \\ -3h_\xi \gamma_i & ((j,k) = (i,i)), \end{cases}$$

$$d_{ijk}^{(5)} = -h_{\xi\eta} c_{0i} \int H\phi_{0i}' \phi_{0j}' \phi_{0k}' \, \mathrm{d}\xi.$$

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